On the scaling limit of finite vertex transitive graphs with large diameter

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Abstract

Let (X_n) be an unbounded sequence of finite, connected, vertex transitive graphs with bounded degree such that $|X_n| = o(\mathrm{diam}(X_n)^q)$ for some $q > 0$. We show that up to taking a subsequence, and after rescaling by the diameter, the sequence (X_n) converges in the Gromov Hausdorff distance to a torus of dimension $< q$, equipped with some invariant length metric. The proof relies on a recent quantitative version of Gromov's theorem on groups with polynomial growth obtained by Breuillard, Green and Tao. If X_n is only roughly transitive and $|X_n| = o\big(\mathrm{diam}(X_n)^\delta\big)$ for $\delta > 1$ sufficiently small, we are able to prove, this time by elementary means, that (X_n) converges to a circle.

1 Introduction

A graph X is *vertex transitive* if for any two vertices u and v in X, there is an automorphism of X mapping u to v. Let (X_n) be a sequence of finite, connected, vertex transitive graphs with bounded degree. Rescale the length of the edges of X_n by $\frac{1}{\mathrm{diam}(X_n)}$, where $\mathrm{diam}(X_n)$ denotes the graph diameter, and denote the resulting metric space by X'_n. A metric space $\mathcal{M}$ is the *scaling limit* of (X_n) if (X'_n) converges to $\mathcal{M}$ in the Gromov Hausdorff distance. See e.g. [4, 7] for background on scaling limits and Gromov Hausdorff distance.

In this paper we address the following questions: when is the scaling limit of such a sequence a compact homogeneous manifold? And in that case, what can be said about the limit manifold? By a compact homogeneous manifold, we mean a compact topological manifold M equipped with a geodesic distance (not necessarily Riemannian), such that the isometry group acts transitively on M. The limit manifold (when it exists) was characterized long ago by Turing, who proved that the only compact Lie groups approximable by finite metric groups are tori [12]. A standard argument allows us to deduce from this that any compact homogeneous manifold approximated by finite homogeneous metric spaces must be a torus (see Proposition 5.1.1).

Our main result is the following.

Theorem 1. *Let (X_n) be a sequence of vertex transitive graphs with bounded degree such that $|X_n| \to \infty$ and $|X_n| = o(\mathrm{diam}(X_n)^q)$. Then (X_n) has a subsequence whose scaling limit is a torus of dimension $< q$ equipped with some invariant proper length metric.*

Remark: We conjecture that the conclusion still holds without the assumption of bounded degree.

Let us consider an example of a sequence for which Theorem 1 holds. Given a ring A, one can consider the Heisenberg group $H(A)$ of 3 by 3 upper unipotent matrices with coefficients in A. Now let X_n be the Cayley graphs of the groups $H(\mathbb{Z}/n\mathbb{Z})$ equipped with the finite generating set consisting of the 3 elementary unipotent matrices and their inverses. The cardinality of X_n equals n^3 and easy calculations shows that its diameter is in $\Theta(n)$. For every n, we have a central exact sequence

$$1 \to \mathbb{Z}/n\mathbb{Z} \to H(\mathbb{Z}/n\mathbb{Z}) \to (\mathbb{Z}/n\mathbb{Z})^2 \to 1,$$

whose center is quadratically distorted. In other words, the projection from X_n to the Cayley graph of $(\mathbb{Z}/n\mathbb{Z})^2$ has fibers of diameter $\simeq \sqrt{n}$. It follows that the rescaled sequence (X_n') converges to a 2-torus.

The proof of Theorem 1 makes crucial use of a recent quantitative version of Gromov's theorem obtained by Breuillard, Green and Tao [3], allowing us to reduce the problem from vertex transitive graphs to Cayley graphs of nilpotent groups. We think an interesting and potentially very challenging open question is to provide a proof of Theorem 1 that does not use this heavy machinery.

In Section 3, we present an elementary proof of a similar result, where the requirement that X_n be Cayley graphs is weakened, but the assumption on the diameter is strengthened. Recall that for $C \geq 1$ and $K \geq 0$, a (C, K)-quasi-isometry between two metric spaces X and Y is a map $f : X \to Y$ such that

$$C^{-1}d(x, y) - K \leq d(f(x), f(y)) \leq Cd(x, y) + K,$$

and such that every $y \in Y$ is at distance at least K from the range of f. Let us say that a metric space X is (C, K)-roughly transitive if for every pair of points $x, y \in X$ there is a (C, K)-quasi-isometry sending x to y. Let us call a family of metric spaces roughly transitive if there exist some $C \geq 1$ and $K \geq 0$ such that each member of the family is (C, K)-roughly transitive. In Section 3, we will provide an elementary proof of the following theorem.

Theorem 2. *Suppose (X_n) is a roughly transitive sequence of finite graphs such that $|X_n| \to \infty$. There exists a constant $\delta > 1$ such that if*

$$|X_n| = o(\mathrm{diam}(X_n)^\delta),$$

then the scaling limit of (X_n) is S^1.

Modifying slightly the proof of Theorem 2, one can prove that for an infinite, roughly transitive graph X with bounded degree, there exists $R > 0$, $C > 0$ and $\delta > 0$ such that if the volume of a ball of radius $R' \geq R$ is less than $CR'^{1+\delta}$, then X is quasi-isometric to $\mathbb{R}$. Even for vertex-transitive graphs, this provides a new elementary proof (compare [5]).

Organization: In Section 2 we prove Theorem 1. In Section 3, we prove Theorem 2. In Section 4, we provide a second elementary proof of Theorem 2, restoring the assumption that the X_n are vertex transitive, rather than quasi-transitive. Finally in the last section, we discuss the optimality of our results and related open questions.

2 Proof of Theorem 1

Given a group G with finite generating set S, we will let (G, S) denote the corresponding Cayley graph. Let D_n denote the diameter of X_n.

2.1 Reduction to nilpotent groups

The first (and main) step of the proof is the following reduction.

Proposition 2.1.1. *There exists a sequence of nilpotent groups N_n with uniformly bounded step and generating sets T_n of uniformly bounded cardinality such that X_n is $(O(1), o(D_n))$-quasi-isometric to (N_n, T_n).*

Fix some $x \in X_n$. Let G_n be the automorphism group of X_n, H_n the stabilizer of x, and S_n be the subset of G_n containing all g such that $g(x)$ is a neighbor of x. When it is clear from context, let G_n denote the Cayley graph of G_n with generating set S_n. We will need the following lemmas.

Lemma 2.1.1 (Doubling in X_n)**.** *There exists a K depending only on q so that the following holds. For all $R_0 > 0$ there exists an $R = R(n) > R_0$ so that $|B_{X_n}(100R)| \leq K|B_{X_n}(R)|$ for all large enough n, and $R(n) = o(D_n)$.*

Proof. Suppose $|B_{X_n}(100R)| > K|B_{X_n}(R)|$ for all $R_0 < R < D_n^{1/2}$. Then

$$|X_n| \geq |B_{X_n}(D_n^{1/2})| > K^{\log_{100}(D_n^{1/2}/R_0)}|B_{X_n}(R_0)| \geq CD_n^{(1/2)\log_{100} K},$$

for some C independent of n. Letting $K = 2^{q+1}$, there is some N such that for $n > N$ this cannot hold. □

Lemma 2.1.2 (Doubling in G_n)**.** *There exists a K depending only on q so that the following holds. For all $R_0 > 0$ there exists an $R = R(n) > R_0$ so that $|B_{G_n}(100R)| \leq K|B_{G_n}(R)|$ for all large enough n, and $R(n) = o(D_n)$.*

Proof. The vertices of X_n correspond naturally to the cosets of H_n, and X_n is isomorphic to the Schreier graph $(G_n/H_n, S_n)$. The projection mapping (G_n, S_n) to $(G_n/H_n, S_n)$ is a graph homomorphism, as $H_n S_n H_n = S_n$, and so for all $h, h' \in H$ we have $y = zs$ for some $s \in S_n$ if and only if $yh = zh's'$ for some $s' \in S$. The homomorphism from (G_n, S_n) to $(G_n/H_n, S_n)$ sends $B_{G_n}(r)$ to $B_{X_n}(r)$ for all $r \geq 0$. So both sides in Lemma 2.1.1 are multiplied by the same constant. $\square$

The main tool in our proof is the following theorem from [3] (although not exactly stated this way in [3], it can be easily deduced from [3, Theorem 1.6] using the arguments of the proof of [3, Theorem 1.3]).

Theorem 2.1.1 (BGT)**.** *Let $K \geq 1$. There is some $n_0 \in \mathbb{N}$, depending on K, such that the following holds. Assume G is a group generated by a finite symmetric set S containing the identity. Let A be a finite subset of G such that $|A^5| \leq K|A|$ and $S^{n_0} \subset A$. Then there is a finite normal subgroup $F \lhd G$ and a subgroup $G' \subset G$ containing F such that*

- *G' has index $O_K(1)$ in G*
- *$N = G'/F$ has step and rank $O_K(1)$.*
- *F is contained in $A^{O_K(1)}$.*

To apply Theorem 2.1.1, we will need the following two lemmas.

Lemma 2.1.3. *(Finite index subgraph) Let X be a connected graph of degree d, and G a group acting transitively by isometries on its vertex set. Let $G' < G$ be a subgroup of index $m < \infty$. Let X' be a G'-orbit. Then X' is the vertex set of some G'-invariant graph that is $(O(m), O(m))$-QI to X, and whose degree is bounded by d^{2m+1}.*

Proof. Denote by $[X']_k$ the k-neighborhood of X' in X. Since it is G'-invariant, $[X']_k$ is a union of G'-orbits of X. But X is a union of m G'-orbits, so $X = [X']_m$. Define a G'-invariant graph on X' by adding edges between two vertices of X' if they are at distance at most $2m+1$. Let $y, y' \in X'$ and let $y = x_0, \ldots, x_{v+1} = y'$ be a shortest path between them in the graph X. Let $y_0 = y$ and $y_{v+1} = y'$, and for each $i = 1 \ldots v$, one can find an element y_i in X' at distance at most m from x_i. Clearly the distance between two consecutive y_i is at most $2m+1$, so they are connected by an edge. Thus, $d_{X'}(y, y') \leq d_X(y, y')$. Since $d_X(y, y') \leq (2m+1) d_{X'}(y, y')$, we see that X' is $(2m+1, m)$-QI to X. $\square$

We will need the following basic lemma. Let $C^j(G)$ be the descending central series of G, i.e. let $C^0(G) = G$, and $C^{i+1}(G) = [G, C^i(G)]$.

Lemma 2.1.4. *Let G be an l-step nilpotent group, and let S be a symmetric subset of G. Then for every $h \in G$ and every $g \in \langle S \rangle$, $[g, h]$ can be written as a product of iterated commutators of the form $[x_1, [\ldots, x_i] \ldots]$, with $i \leq l-1$, where for each $j = 1 \ldots i$, $x_j \in S \cup \{h^{\pm}\}$, and at least one of the $x_j \in \{h^{\pm}\}$.*

Proof. Without loss of generality we can suppose that G is the free nilpotent group of class l generated by $S \cup \{h^{\pm}\}$. We shall prove the lemma by induction on l. For $l = 1$, the statement is obvious as the group G is abelian. The case $l = 2$ is interesting as it reveals the key idea: one can use the formula $[h, gg'] = [h, g][h, g']$ to break $[h, g]$ into a product of $[h, s]$, where $s \in S$.

Let us assume that the statement is true for $l \leq l_0$ and suppose that G is $(l_0 + 1)$-step nilpotent. Applying the induction hypothesis to $G/C^{l_0}(G)$, we deduce that $[g, h]$ is the product of some $y \in C^{l_0}(G)$ with elements $[x_1, [\ldots, x_i] \ldots]$ where $i \leq l_0 - 1$, each $x_j \in S \cup \{h^{\pm}\}$, and at least one x_j in each term is in $\{h^{\pm}\}$.

The element y can be written as a product of elements of the form $[g_1, [\ldots, g_{l_0}] \ldots]$ with $g_j \in G$. Since G is (l_0+1)-step nilpotent, the iterated commutator $[g_1, [\ldots, g_{l_0}] \ldots]$ induces a morphism of abelian groups $\bigotimes_{i=1}^{l_0} G \to C^{l_0}(G)$. Writing each g_j as a word in $S \cup \{h^{\pm}\}$ and using this morphism, we can write y as a product of terms of the form $[a_1, [\ldots, a_{l_0}] \ldots]$ with $a_i \in S \cup \{h^{\pm}\}$.

We have shown that $[g, h]$ can be written as a product of iterated commutators in $S \cup \{h^{\pm}\}$; it remains to show that each term from y contains at least one $x_j \in \{h^{\pm}\}$. The terms obtained from y commute with each other, so we can gather the terms without $h^{\pm}$ into a single word w. Let N denote the normal subgroup generated by h. Since $[g, h] = h^g h^{-1}$, we know that $[g, h] \in N$. Similarly, each iterated commutator containing $h^{\pm}$ is in N, so $w \in N$. But w is also in the subgroup H generated by S, and because G is the free nilpotent group of class $(l_0 + 1)$ generated by $S \cup \{h^{\pm}\}$, we have that $N \cap H$ is trivial. Thus w is trivial. This completes the proof of the lemma. □

Corollary 2.1.1. *Let G a l-step nilpotent group generated by some symmetric set S. Then for every element $h \in G$, the normal subgroup generated by h is generated as a subgroup by the elements h^x, where $x \in (S \cup \{h^{\pm}\})^k$, with $k \leq 4^l$.*

Proof. Note that $h^g = [g, h]h$. Applying the lemma to the commutator $[g, h]$ yields a product of iterated commutators with letters in $S \cup \{h^{\pm}\}$, where $h^{\pm}$ appears at least once in each. We leave to the reader to check that such a commutator is a product of conjugates of $h^{\pm}$ by elements x whose word lengths with respect to $S \cup \{h^{\pm}\}$ are at most that of an iterated commutator $[a_1, [\ldots, a_{l-2}] \ldots]$. The length k_l of such commutator is defined inductively as $k_1 = 0$, and $k_{l+1} = 2k_l + 2$. Thus, $k_l \leq 4^l$. □

The following lemma has its own interest (compare [6]).

Lemma 2.1.5. *Let X be a transitive graph with degree $d < \infty$, and let G be a nilpotent group acting faithfully and transitively on X. Then the cardinality of any vertex stabilizer H is in $O_{d,r,l}(1)$, where r and l are the rank and step of G. More precisely, let $H = \oplus_p H_p$ be its p-torsion decomposition (which holds since H is nilpotent). Then $\max_p p^n \leq d^{8^l}$, where p^n is the maximal order of an element of H_p.*

Proof. Recall that the rank of any subgroup of G is in $O_{r,l}(1)$, hence the second part of the lemma implies the first one.

The set of vertices of X can be identified with G/H, with x corresponding to the trivial coset H. Faithfulness of the action is equivalent to the fact that H does not contain any non-trivial normal subgroup of G. Observe that the action by left-translation by H preserves the set of neighbors of H. As in the proof of Lemma 2.1.2, we equip G with a generating set S such that X is isomorphic to the Schreier graph $(G/H, S)$, which implies that S is bi-H-invariant. In particular one has $H \subset S^2$.

The proof roughly goes as follows: for every element of H_p of order p^n, we will show that there is a vertex in the ball of radius 8^l of X whose orbit under the action of H has cardinality at least p^n.

By assumption, the normal subgroup generated by $h^{p^{n-1}}$ is not contained in H. Thus by Corollary 2.1.1, there exists $g \in (S \cup H)^{4^l} \subset S^{8^l}$ such that $g^{-1}h^{p^{n-1}}g$ does not belong to H. But then this implies that for all $i = 1, \ldots, p^n$, the vertices $h^i gH$ are distinct, for if $h^i gH = h^j gH$ for some $1 \leq i < j \leq p^n$, then $y = g^{-1}h^{i-j}g \in H$. Write $i - j = p^a b$, with $a < n$ and b coprime to p. There is some c so that $cb = 1 \mod p^n$. Then H contains $y^{cp^{n-1-a}} = g^{-1}h^{p^{n-1}}g$, which is not in H, giving a contradiction. Hence, the $h^i gH$ are distinct for every $1 < i < j \leq p^n$. Since g has length at most 4^l, and the left-translation by H preserves the distance to the origin, the number of such translates is at most the cardinality of the ball of radius 8^l, i.e. at most d^{8^l}, so we are done.

Observe that the same argument can be used to prove that H does not contain any element of infinite order. $\square$

Proof of Proposition 2.1.1. By Lemma 2.1.2, we can find a sequence R_n with both R_n and D_n/R_n tending to infinity such that $|B_{G_n}(100R_n)| \leq K|B_{G_n}(R_n)|$. Then by Theorem 2.1.1 applied to $A = B_{G_n}(R_n)$, we obtain a sequence of groups G'_n and $N'_n = G'_n/F_n$ such that G'_n has uniformly bounded index in G_n, F_n has diameter $o(D_n)$, and N'_n is nilpotent with uniformly bounded step and rank. Associate the generating set S'_n (to be defined) with G'_n, and the projection T'_n of S'_n to N'_n.

Proposition 2.1.1 now results from the following facts.

- Since G'_n has bounded index in G_n, by Lemma 2.1.3 there exists a G'_n-invariant graph structure X'_n on G'_n/H'_n which is $(O(1), O(1))$-QI to X_n and has bounded degree. Define S'_n to be the generating subset of G'_n consisting of elements projecting to $B(H, 1)$ in X'_n.

- The graph Y_n obtained by quotienting X'_n by the normal subgroup F_n also has bounded degree, and the quotient map $X'_n \to Y_n$ has fibers of diameter $o(D_n)$. Hence Y_n is $(1, o(D_n)$-QI equivalent to X'_n.

- Let L_n be the kernel of the action of N'_n on Y_n, and define $N_n = N'_n/L_n$. Then N_n acts faithfully and transitively on Y_n, so Y_n is isomorphic to $(N_n/H''_n, T_n)$, where H''_n is the stabilizer of a vertex x, and T_n is the set of elements of N_n taking x to a neighbor of x. But N_n and therefore H'' is nilpotent of bounded step and rank and Y_n has bounded degree, so we deduce from Lemma 2.1.5 that H''_n is uniformly bounded. Thus, Y_n is $(O(1), O(1))$ quasi-isometric to (N_n, T_n), and T_n is uniformly bounded, and N_n inherits the uniform bound on step and rank from N'_n.

Recapitulating, we constructed quasi-isometries between X_n and X'_n, then between X'_n and Y_n, and finally between Y_n and the Cayley graph of N_n. The multiplicative constants of these quasi-isometries are all bounded, and the additive ones are bounded by $o(D_n)$, so the proposition is proved. □

2.2 Existence of the limit

By Proposition 2.1.1, it now suffices to show that (N_n, T_n) converges for the Gromov-Hausdorff metric to a torus of dimension at most $q-1$. In this section, we will focus on the convergence to some Lie group.

Proposition 2.2.1. *Suppose (N_n) is a sequence of step l finite nilpotent groups with bounded generating sets T_n. Then for every $D_n \to \infty$, a subsequence of the Cayley graphs $(N_n, d_{T_n}/D_n)$ converges in Gromov-Hausdorff distance to a connected nilpotent Lie group, equipped with some invariant proper length metric.*

Note that in Proposition 2.2.1, we do not require D_n to be the diameter of N_n, but rather any unbounded sequence. Specialised in the case where the groups N_n are finite, and D_n is the diameter of the Cayley graph (N_n, T_n), this proposition implies that the limit is a finite dimensional torus (since a compact connected nilpotent Lie group is necessarily a torus).

We will use the following theorem from [10]:

Theorem 2.2.1. *Let N the free c-step nilpotent group on r generators, with its standard generating set S, and $D_n \to \infty$. Then $(N, d_S/D_n)$ converges to some simply connected nilpotent Lie group $N_{\mathbb{R}}$ equipped with some left-invariant proper length metric*[1] *d'.*

Proof of Proposition 2.2.1. The bound on the step and on the cardinality of the generating set ensures that the sequence (N_n, T_n) has a uniform doubling constant (since it can be seen as a quotient of some fixed nilpotent group: the free rank r step l nilpotent group, with $r, l = O(1)$). Hence the rescaled sequence is relatively compact for the Gromov-Hausdorff metric [7]. Therefore up to passing to a subsequence, we can suppose that the sequence converges so some limit space (X, d).

Recall that if a sequence of metric spaces (Y_n, d_n) converges to some locally compact space for the Gromov-Hausdorff metric (Y, d), then for any ultra-filter on $\mathbb{N}$, the corresponding ultra-limit of (Y_n, d_n) is naturally isometric to (Y, d) [7]. Given a sequence $y_n \in Y_n$, let $[y_n]$ denote the equivalence class of (y_n) in the ultralimit.

Let $p_n : (N, d_S/D_n) \to (N_n, d_{T_n}/D_n)$ be the natural projection. The p_n are each 1-Lipschitz and surjective, so there is a projection $p : (N_{\mathbb{R}}, d') \to (X, d)$ from the limit $(N_{\mathbb{R}}, d')$ of the $(N, d_S/D_n)$ to the limit (X, d) of the $(N_n, d_{T_n}/D_n)$ such that for each sequence (x_n) in N,

$$[p_n(x_n)] = p([x_n]).$$

We also have that for every $g \in N_n$, there is a $x \in N$ so that $p_n(x) = g$ and $|g|_{T_n} = |x|_S$.

[1] More precisely some Carnot-Caratheodory metric

We claim that X is naturally a group. More precisely if $dist(g_n, g'_n) \to 0$ and $dist(h_n, h'_n) \to 0$ then $[g_n h_n] = [g'_n h'_n]$. We can write $g'_n = g_n a_n$ and $h'_n = h_n b_n$ with $|a_n|, |b_n| \to 0$. We can choose $x_n, c_n, y_n, d_n \in (N, d/D_n)$ that are mapped by p_n to g_n, a_n, h_n, and b_n, respectively, and so that $|c_n|, |d_n| \to 0$ in $(N, d/D_n)$. Because $(N, d/D_n)$ is a group, we have $[g_n h_n] = [p(x_n)p(y_n)] = [p(x_n y_n)] = p[x_n y_n] = p([x_n a_n y_n b_n]) = [p(x_n a_n y_n b_n)] = [g_n a_n h_n b_n] = [g'_n h'_n]$. ☐

2.3 End the proof of Theorem 1: bound on the dimension of the limiting torus

In the last section, we established in that (X_n) converges (up to subsequence) to some torus T equipped with some proper invariant length metric d_∞. The only thing that remains to be proved is the bound on the dimension.

Proposition 2.1.1 reduced to the case where X_n are Cayley graphs of nilpotent groups with bounded generating set and step. We will now reduce the problem to the case of a sequence of Cayley graphs (A_n, U_n) where A_n is abelian and U_n has bounded cardinality. In the last section we proved that the limit is an *abelian* compact Lie groups. We did not prove it directly, but rather using the well-known fact that connected nilpotent compact Lie groups are abelian. Hence it suggests that the sequence N_n has the same limit as its abelianization. This is indeed true, and relies on the following simple lemma.

Lemma 2.3.1. *Let G be a step l-step nilpotent group. Then every element x in $[G, G]$ can be written as a product of l commutators.*

Proof. The statement is easy to prove by induction on l. There is nothing to prove if $l = 1$, so let us assume that $l > 1$. Note that $C^l(G) = \{1\}$, so that $C^{l-1}(G)$ is central. Let $x \in [G, G]$. By induction, x can be written as a product of $l - 1$ commutators times an element of $C^{l-1}(G)$. Hence it is enough to prove that every element of $C^{l-1}(G)$ can be written as a single commutator. Recall that the iterated commutator $[x_1, [x_2[\ldots, x_l]\ldots]$ induces a morphims from $\bigotimes_{i=1}^l A$ to $C^{l-1}(G)$, where $A = G/[G, G]$. In particular its range is a subgroup of $C^{l-1}(G)$. Since it contains generators of $C^{l-1}(G)$ it is equal to $C^{l-1}(G)$, which proves the lemma. ☐

Corollary 2.3.1. *Let (T, d_∞) be the scaling limit of (N_n, T_n), and let $\pi_n : N_n \to A_n = N_n/[N_n, N_n]$ be the projection on the abelianization. Then $(A_n, \pi_n(T_n))$ converges to (T, d_∞).*

Proof. The only thing to be checked is that any sequence $g_n \in [N_n, N_n]$ converges to the neutral element of T. But by Lemma 2.3.1, g_n can be written as a bounded product of commutators, each of which converges to a commutator in the ultra limit (see proof of Proposition 2.2.1). So the conclusion results from the fact that T is abelian. ☐

Before stating the main result of this section, let us introduce a useful definition.

Definition: Let (A, U) be a Cayley graph where A is abelian, $U = \{\pm e_1, \ldots, \pm e_k\}$. Define its radius of freedom $R_f(A, U)$ to be the largest R such that the natural projection $\mathbb{Z}^k \to A$ is isometric in restriction to the ball of radius R.

The following proposition concludes the proof of Theorem 1:

Proposition 2.3.1. *Let (A_n, V_n) be a sequence of (finite) Cayley graphs where A_n is abelian, $V_n = \{\pm e_1, \dots, \pm e_k\}$ for some fixed k. Then up to some subsequence, the rescaled sequence (A_n, V_n) converges to a torus T of dimension at most the largest integer j such that $D_n^j = O(|A_n|)$.*

Proof. Let d denote the dimension of T; we want to show $d \leq j$. Clearly we have $d \leq k$. The strategy is to show that we can reduce the number of generators (changing also the group) to precisely $2d$ such that the limit still has at least the same dimension.

- Claim 1: Let $R_f = R_f(A, V)$. By definition, $R_f + 1$ is the smallest integer such that there exist $(n_1, \dots, n_k)$ with $\sum |n_i| \leq 2(R_f + 1)$ such that $n_1 e_1 + \dots + n_k e_k = 0$. Up to permuting the generators, we can assume that $n_k \neq 0$.
- Claim 2: Let B be the subgroup of A generated by the set $V = \{\pm e_1, \dots, \pm e_{k-1}\}$. The map $(B, V) \to (A, U)$ is a graph morphism and therefore is 1-Lipschitz, and any element in A lies at distance at most R_f from its image.
- Claim 3: If $R_f(A_n, U_n) = o(D_n)$, then the sequence of (B_n, V_n) rescaled by D_n converges to a connected abelian Lie group surjecting to the limit of (A_n, U_n).
- Claim 4: Suppose that $R_f(B_n, V_n) = o(D_n)$, then Claim 1 to Claim 3 still hold so that we obtain a sequence (C_n, W_n) where C_n is the subgroup generated a symmetric subset W comprising $k-2$ elements (and their inverses) of V, and such that the limit surjects to the limit of (B_n, Vn).

Iterating this until R_f becomes comparable (up to some subsequence) to D_n, we obtain that for some $l \leq k$, and up to reindexing the e_j, the Cayley subgraphs generated by $(\pm e_1, \dots \pm e_l)$, rescaled by D_n converge to a torus of dimension at least d: hence $d \leq l$. But R_f being comparable to D_n, the volume of this subgraph is at least of the order of R_f^l. Since it is a subgraph, we have $l \leq j$. Hence we duly have $d \leq j$.

□

3 Proof of Theorem 2

In this section, we will provide an elementary proof of Theorem 2.

3.1 Proof outline

The proof of Theorem 2 goes approximately as follows. We show using rough transitivity that if the graph does not converge to a circle, then it must contain a caret of size proportional to the diameter. Then, iterating using rough transitivity, we generate large volume, contradicting the assumption.

3.2 Proof

First, we will define a few key terms.

Definition: Given $K \geq 0$ and $C \geq 1$, a (C, K)-*quasi-geodesic* in a metric space X is a (C, K)-quasi isometrically embedded copy of the interval $[1, k]$ into X; i.e. a sequence of points $x_1, \dots x_k \in X$ such that

$$C^{-1}(j-i) - K \leq d(x_i, x_j) \leq C(j-i) + K$$

for all $1 \leq i < j \leq k$.

Let $B(v, r)$ denote the ball of radius r around a vertex v. A quasi-caret of radius $\geq R$ consists of three quasi-geodesic segments γ_1, γ_2 and γ_3 that start from a point v_0, escape from $B(v_0, R)$, and move away from one another at "linear speed." In other words,

Definition: A *quasi-caret of radius R* is a triple γ_1, γ_2, γ_3 of quasi-geodesics from a vertex v_0 to vertices v_1, v_2, and v_3, respectively, such that $d(v_0, v_i) = R$ for $i = 1, 2, 3$, and there is a constant $c > 0$ satisfying that for all k_1, k_2, k_3, $d(\gamma_i(k_i), \gamma_j(k_j)) \geq c \max\{k_i, k_j\}$ for $i \neq j$.

If a roughly transitive graph has a quasi-caret of radius $\geq \epsilon D$, then by moving around this caret with quasi-isometries (with uniform constants), we obtain at every point of the graph a quasi-caret (with uniform constants) of radius $\geq \epsilon' D$.

The proof of Theorem 2 follows from the following four lemmas.

Lemma 3.2.1. *Let $D = \mathrm{diam}(X)$. Suppose there exists a quasi-caret of radius $R = \epsilon D$ for some $\epsilon > 0$ in a finite (C,K)-roughly transitive graph X. Then $|X| \geq \epsilon' D^\delta$, where ϵ' and $\delta > 1$ depend only on C, K and ϵ.*

Proof. To avoid complicated expressions that would hide the key idea, we will remain at a rather qualitative level of description, leaving most calculations to the reader.

Given a quasi-caret $(\gamma_1, \gamma_2, \gamma_3)$, we can stack a sequence of disjoint consecutive balls along the γ_j's, whose radii increase linearly with the distance to the center v_0. More precisely, one can find for every $j = 1, 2, 3$ a sequence of balls $B^j_k = B(\gamma_j(i_k), r^j_k)$ such that

- $C^{-1} d(v_0, \gamma(i_k)) \leq r^j_k \leq C d(v_0, \gamma(i_k))$ for some constant $C \geq 1$,
- $r^j_k \geq c' R$ for some $0 < c' < 1$ independent of R,
- $\sum_{j,k} r^j_k \geq \alpha R$ where $\alpha > 1$ is also independent of R,
- the distance between B^j_k and B^j_{k+1} equals 1, and all these balls are disjoint (when j and k vary).

Now, fix some $R \leq \epsilon D$, and consider a ball $B(x, R)$. It contains a quasi-caret of radius R. This caret can be replaced by the balls described above, each one of them containing a quasi-caret of radius r^j_k. The sum of the radii of these carets is at least αR. We can iterate this procedure within each ball, so that at the k-th iteration, we obtain a set of disjoint balls in $B(x,R)$ whose radii sum to at least $\alpha^k R$. Because the radius R decreases by a factor no smaller than c' each time, we can iterate $\log_{1/c'} R$ times. After $\log_{1/c'} R$ iterations, each ball still has positive radius, so $|B(x,R)| \geq \alpha^{\log_{1/c'} R} R = R^{1+\log_{1/c'}(\alpha)}$. Since $\log_{1/c'}(\alpha) > 0$, this proves the lemma. □

Next we will show that under the assumption that no such caret exists, our graphs locally converge to a line. More precisely,

Lemma 3.2.2. *Let X_n be a roughly transitive sequence of graphs of diameter D_n going to infinity whose carets are of length $o(D_n)$. Then there exists $c > 0$ such that for n large enough, any ball of radius cD_n is contained in the $o(D_n)$-neighborhood of a two-sided geodesic line.*

Proof. By rough transitivity, it is enough to prove the lemma for some specific ball of radius cD_n. Start with a geodesic $[x,y]$ of length equal to the diameter D_n. Let z be the middle of this geodesic. We are going to show that the ball of radius $D_n/10$ around z is contained in a $o(D_n)$-neighborhood of $[x,y]$. If this was untrue, we would find a constant c' such that $B(z, D_n/10)$ contains an element w at distance at least $c'D_n$ of $[x,y]$. Now pick an element z' in $[x,y]$ minimizing the distance from w to $[x,y]$. The shortest path from z' to w, together with the two segments of the geodesic starting from z' form a caret of size proportional to D_n. □

If we knew that X_n converges, and that the limit is homogenous and compact, then this lemma would show that the limit is a locally a line, and thus must be S^1. If we knew that there was a large geodesic cycle in X_n, then Lemma 3.2.1 would show that all vertices are close to the cycle, which would also imply that the limit is S^1. However, we know neither of these two facts a priori, so the next lemma is necessary to complete the poof.

Lemma 3.2.3. *Suppose X_n has the property that for some $c > 0$, and for n large enough, any ball of radius cD_n is contained in the $o(D_n)$-neighbourhood of a two-sided geodesic line. Then its scaling limit is S^1.*

Proof. Let $x_1, ..., x_k$ be a maximal $cD_n/10$-separated set of points of X_n, and let B_j be the corresponding balls of radius $cD_n/100$.

We consider the graph H_n whose vertices are labeled by the balls B_j and such that two vertices are connected by an edge if the corresponding balls are connected by a path avoiding the other balls.

By maximality, for any $v \in X_n$, there is at least on x_i in $B(v, cD_n/5)$. Let us consider a fixed x_j. In $B(x_j, cD_n)$, X_n is well approximated by a line, so there are vertices v_1 and v_2 on either side of x_j such that the balls of radius $cD_n/5$ around v_1, v_2, and x_j are disjoint. Thus, there must be an x_i on either side of x_j. Picking on each side the x_i that is closest to x_j, we see that the corresponding balls are connected in H_n. Moreover since removing these two balls disconnects the ball $B(x_j, cD_n/100)$ from all other x_i's, we see that the degree of

H_n is exactly two. Hence the graph H_n is a cycle that we will now denote by $\mathbb{Z}/k\mathbb{Z}$. To simplify notation, let us reindex the balls B_j accordingly by $\mathbb{Z}/k\mathbb{Z}$.

Let us show that the graph X_n admits a (simplicial) projection onto $\mathbb{Z}/k\mathbb{Z}$. If we remove the balls B_i, we end up with a disjoint union of graphs $C_1, \dots, C_k$, such that C_j is connected to B_j and B_{j+1}. Let $V_j = B_j \cup C_j$. The graph V_j connects to and only to V_{j-1} and V_{j+1}. Hence we have a projection from X_n to the cyclic graph $\mathbb{Z}/k\mathbb{Z}$ sending V_j to the vertex j, and edges between V_j and V_{j+1} to the unique edge between j and $j+1$.

Recall that the distance between two consecutive balls B_i and B_{i+1} is at least $cD_n/20$. Now take a shortest loop $\gamma = (\gamma(1), \dots, \gamma(m) = \gamma(0))$ in X_n among those projecting to homotopically non trivial loops in the graph associated to $\mathbb{Z}/k\mathbb{Z}$. Clearly this loop has length at least $ckD_n/20$ (since it passes through all balls B_i). We claim moreover that it is a geodesic loop. Without loss of generality, we can suppose that $\gamma(0)$ starts in B_0 and that the next ball visited by γ after B_0 is B_1. Observe that although γ might exit some B_i and then come back to it without visiting any other B_j in the meantime, it *cannot* visit B_i, then go to B_{i+1}, and then back to B_i (without visiting other balls in the meantime). Indeed such a bactrack path could be replaced by a shorter path staying within B_i, contradicting minimality. It follows that the sequence of B_i's visited by γ (neglecting possible repetitions) is given by $B_0, B_1 \dots B_k = B_0$; namely it corresponds to the standard cycle in $\mathbb{Z}/k\mathbb{Z}$. The same argument implies that the sequence of balls visited by a any geodesic joining two points in X_n corresponds to a (possibly empty) interval in $\mathbb{Z}/k\mathbb{Z}$.

Now, suppose for sake of contradiction that γ is not geodesic. This means that there exists an interval of length $\leq m/2$ in γ which does not minimize the distance between its endpoints. But then applying the previous remark, we see that replacing either this interval or its complement by a minimizing geodesic yields a loop whose projection is homotopically non-trivial, hence contradicting our minimal assumption on γ.

We therefore obtain a geodesic loop in X_n whose Hausdorff distance to X_n is in $o(D_n)$. Hence the scaling limit of X_n exists and is isometric to S^1. $\square$

4 A second elementary proof

In this section, we present a second elementary proof of Theorem 2, but under the stronger assumption that the X_n are vertex transitive. This proof strategy gives us $\delta = 1 - \frac{1}{\log_3(4)}$.

Theorem 4.0.1. *Suppose X_n are vertex transitive graphs with $|X_n| \to \infty$ and*

$$|X_n| = o(\operatorname{diam}(X_n)^{2-\frac{1}{\log_3(4)}}.$$

Then the scaling limit of (X_n) is S^1.

4.1 Proof outline

To prove Theorem 4.0.1, It suffices to show that in a finite vertex transitive graph with small volume relative to its diameter, there is a geodesic cycle whose length is polynomial in the diameter. If a long caret is rooted at a vertex on this cycle, then using transitivity and iteration we generate large volume, contradicting the assumption. Thus, all vertices must be close to the cycle.

To find a large geodesic cycle, we use the fact that a finite vertex transitive graph X contains a $\mathrm{diam}(X)/8$-fat triangle. If this triangle is homotopic to a point after filling in small faces, this will imply large area and will violate our assumption. Thus, there is a loop that is not contractable. The smallest non-contractable loop is a geodesic cycle, and since we filled in all small cycles, this geodesic cycle must be large.

4.2 Proof

We begin by proving a version of Lemma 3.2.1 for vertex transitive graphs.

Definition: A *3-caret of branch-length R* is a triple γ_1, γ_2, γ_3 of geodesics from a vertex v_0 to vertices v_1, v_2, and v_3, respectively, such that $d(v_0, v_i) = R$ for $i = 1, 2, 3$, and for all k_1, k_2, k_3, $d(\gamma_i(k_i), \gamma_j(k_j)) \geq \max\{k_i, k_j\}$ for $i \neq j$.

Lemma 4.2.1. *Let $D = \mathrm{diam}(X)$. Suppose there exists a 3-caret of branch length $R = \epsilon D^c$ for some $\epsilon, c > 0$ in a finite vertex transitive graph X. Then $|X| > \epsilon' D^{1+c(\log_3(4)-1)}$, where $\epsilon' = (1/2)\epsilon^{\log_3(4)-1}$.*

Proof. Suppose γ_1, γ_2, and γ_3 form a 3-caret of branch length R. Let u_1, u_2, and u_3 denote the vertices at distance $2R/3$ from v_0 on γ_1, γ_2, and γ_3, respectively, and let $u_0 := v_0$. The u_i are at pairwise distance $2R/3$ from each other, and so $B(u_i, R/3)$ are pairwise disjoint. By vertex transitivity, there is a 3-caret of branch length R centered at each u_i, which intersects $B(u_i, R/3)$ as a 3-caret of branch length $R/3$. Thus, we have four disjoint balls of radius $R/3$, each containing a 3-caret of radius $R/3$.

We can iterate this procedure, dividing R by three at each step and multiplying the number of disjoint balls by four. So for any m, $B(v_0, R)$ contains 4^m balls, each of which contains a 3-caret of branch length $R/3^m$. Letting $m = \log_3(R)$, we have that $B(v_0, R)$ contains 4^m disjoint 3-carets of branch length 1. In particular, $|B(v_0, R)| \geq 4^m = R^{\log_3(4)}$.

There exists a geodesic path γ in X of length D. Let $R = \epsilon D^c$. Then it is possible to take vertices $v_1, \ldots, v_{D/2R}$ in γ such that $B(v_i, R) \cap B(v_j, R) = \emptyset$ for all $i \neq j$. Summing the number of vertices in $B(v_i, R)$ for $1 \leq i \leq D/(2R)$, and using that $|B(v_i, R)| \geq R^{\log_3(4)}$, we have

$$|X| \geq D/(2R) \cdot R^{\log_3(4)} = (1/2)\epsilon^{\log_3(4)-1} D^{1+c(\log_3(4)-1)}.$$

□

The fact that a 3-caret of branch length R implies $|B(v_0, R)| \geq R^{\log_3(4)}$ also has consequences for infinite vertex transitive graphs. For example, if an infinite vertex transitive graph X has linear growth, then there is an upper bound on the size of a 3-caret in X. Since X has a bi-infinite geodesic γ and a vertex at distance R from γ implies a 3-caret of branch length R, every vertex in X must be within a bounded neighborhood of γ. Conversely, if

X does not have linear growth, then there must be vertices at arbitrary distances from any fixed bi-infinite geodesic. Thus X must have growth at least $O(n^{\log_3(4)})$.

Next, we will show that every vertex-transitive graph with a large diameter has a large geodesic cycle. We will use the following theorem from [2].

Definition: A geodesic triangle with sides s_1, s_2, s_3 is δ-fat if for every vertex v in X,

$$dist(v, s_1) + dist(v, s_2) + dist(v, s_3) \geq \delta.$$

Theorem 4.2.1. *Every finite vertex transitive graph with diameter D contains a $(1/8)D$-fat triangle.*

For completeness, here is the short proof. Given vertices u and v, let uv denote a shortest path from u to v

Proof. Suppose X is finite and transitive, and D is its diameter. Let w and z realize the diameter, i.e. $|wz| = D$. By transitivity there is a geodesic path xy that has z as its midpoint and length D. Suppose the triangle wxy is not δ-fat. Then there is a point a on xy such that the distance from a to wy is at most 2δ and the distance from a to wx is at most 2δ. Suppose, w.l.o.g. that a is closer to x than to y. We have $|ax| + |ay| = D$, $|wa| + |ax| < 2\delta + D$ (because a is within 2δ of wx), $|wa| + |ay| < 2\delta + D$. Add these latter two and subtract the previous equality, and get $|wa| < D/2 + 2\delta$. Since $|wz| = D$, this means that $|za| > D/2 - 2\delta$. Since a is on xy and closer to x, this means that $|xa| < 2\delta$. Since a is within 2δ from wy, we have $|wy| > |wa| + |ay| - 2\delta$. Since $|xa| < 2\delta$ and $|xy| = D$ this gives $|wy| > |wa| + D - 4\delta$. Since $|wy|$ is at most D, this implies $|wa| < 4\delta$. But $|za|$ is at most $D/2$. so $D = |wz| \leq |wa| + |za| < 4\delta + D/2$ So $D < 8\delta$. □

Lemma 4.2.2. *Suppose X is a finite d-regular vertex-transitive graph such that $|X| < (\alpha/d)D^{2-c}$, where $\alpha = \sqrt{3}/576$. Then X contains a geodesic cycle of length D^c.*

Proof. We will begin by proving two claims.

Claim 1: Suppose H is a d-regular planar graph, every face of H except the outer face has a boundary of length at most D^c, and H contains a $(1/8)D$-fat geodesic triangle. Then $|H| > (\alpha/d)D^{2-c}$.

This is a variant of Besicovich' lemma for squares. Fill each face f of H with a simply connected surface of area at most $|f|^2$ so that distances in H are preserved (for example, a large portion of a sphere), and consider the $(1/8)D$-fat geodesic triangle in the simply connected surface X obtained. The triangle has sides s_1, s_2, and s_3, of lengths at least $(1/8)D$. The map f from X to $\mathbb{R}^3_+$ taking a point x to $(dist(x, s_1), dist(x, s_2), dist(x, s_3))$ is 3-Lipschitz, so the area of the image of f is smaller than 9 times the area of X. For each (x_1, x_2, x_3) in the image we have $x_1 + x_2 + x_3 > (1/8)D$, so projecting radially to the simplex $x_1 + x_2 + x_3 = (1/8)D$ does not increase the area. The projection of the image of the boundary of the triangle is the boundary of the simplex, so the projection is onto. Thus, the area of X is bigger than $1/9\sqrt{3}((1/8)D)^2$.

Each face f of H contributes $|f|^2$ to the area to X, so we can say that each vertex on the border of f contributes $|f|$ to the area of X. Each vertex of H participates in at most d faces, each of which is of size at most D^c so area$(X) \leq dD^c|H|$. Thus $|H| > \alpha/dD^{2-c}$ where $\alpha = (1/9)\sqrt{3}(1/8)^2$.

Claim 2: Suppose X is a finite d-regular graph that contains a $(1/8)D$-fat triangle A, and let T denote the topological space obtained from X by replacing each cycle of length at most D^c with a euclidean disc whose boundary matches the cycle. If A is homotopic in T to a point, then $|X| > (\alpha/d)D^{2-c}$.

Suppose A is homotopic to a point. Then a continuous map from S^1 to A can be extended to a continuous map from the disk B^1 to $T(X, D^c)$. The image of this map has a planar sub-surface S with boundary A. Intersecting S with X, we obtain a planar subgraph H of X such that each face has a boundary of length at most D^c except for the outer face, which has A as a boundary. By Claim 1, $|X| \geq |H| > (\alpha/d)D^{2-c}$.

Now we will prove the lemma. Because $|X| < (\alpha/d)D^{2-c}$, Claim 2 tells us that T is not simply connected. Any loop in T is homotopic to a loop in X, so since T is not simply connected, there exists a topologically non-trivial loop in X. Let ℓ denote the non-trivial loop in X which has minimal length.

Given any two vertices u and v in ℓ, the shortest path from u to v is homotopic to at most one of the two paths p_0, p_1 in ℓ between u and v; say it is not homotopic to p_0. If the length of p^* is shorter than the minimum length of p_0 and p_1, then it would be possible to replace p_1 with p^* to obtain a loop ℓ' which is non-trivial and shorter than ℓ. Thus, the shortest path in X between any two vertices in ℓ is a path in ℓ, so ℓ is a geodesic cycle.

Because all cycles of length less than D^c are homotopic to a point in T, ℓ must have length greater than D^c. □

Proof of Theorem 4.0.1. Let D_n denote the diameter of X_n and $c = \frac{1}{\log_3(4)}$. It suffices to show that for large enough n, there is a geodesic cycle C_n in X_n such that $|C_n| > {D_n}^c$ and

$$\max_{v \in X_n}(dist(v, C_n)) = o({D_n}^c).$$

For large enough n, $|X_n| < (\alpha/d){D_n}^{2-c}$, so Lemma 4.2.2 guarantees the existence of a geodesic cycle C_n with $|C_n| > {D_n}^c$. By Lemma 4.2.1, if there were a 3-caret of branch length $\epsilon {D_n}^c$, we would have $|X_n| \geq \epsilon' {D_n}^{1+c(\log_3(4)-1)} = \epsilon' {D_n}^{2-c}$. So for all ϵ and large enough n, there is no 3-caret of branch length $\epsilon {D_n}^c$. But for $\epsilon < {D_n}^c/4$, a vertex at distance $\epsilon {D_n}^c$ from C_n implies a 3-caret of branch length $\epsilon {D_n}^c$. Thus, for every ϵ and for large enough n, all vertices in X_n are within distance $\epsilon {D_n}^c$ from C_n. □

5 Further results and open questions

5.1 Compact homogeneous manifolds approximable by finite homogeneous metric spaces are tori

Proposition 5.1.1. *Let M be a homogeneous riemannian manifold which is the Gromov-Hausdorff limit of a sequence of homogeneous finite metric spaces. Then M is a torus.*

Proof. Let (X_n, d_n) be the sequence of approximating metric spaces, and let G_n be their isometry groups. Let $\hat{d}_n$ be the bi-invariant distance on G_n defined by

$$\hat{d}_n(f, g) = \max_{x \in X_n} d_n(f(x), g(x)).$$

Suppose X_n converges to some compact metric space X, we claim that G_n has a subsequence converging to a group G of isometries acting transitively on X. Then the proposition will follow from Turing's theorem mentioned in the introduction. Since the argument is standard, we will only sketch it, and leave the details to the reader. The sequence X_n being convergent, it is equi-relatively compact: for all $\epsilon > 0$, there exists N such that X_n is covered by N balls of radius $\epsilon > 0$. It is then easy to check that the sequence G_n is also equi-relatively compact, which implies that it has a converging subsequence, whose limit will be denoted by G.

For the second part of the proof, one can use the fact that when a sequence of metric spaces Y_n converges to a metric space for the Gromov-Hausdorff metric, then the limit is natually isometric to any ultralimit of Y_n with respect to any (non-principal) ultrafilter β on $\mathbb{N}$. Then seeing X and G as ultralimits of X_n, resp. G_n, it is easy to check that G is a group and acts transitively by isometries on X. □

5.2 What about an analogue of Theorem 2 in higher dimensions?

Theorem 2 cannot be generalized to "higher" dimensions (as in Theorem 1) because any compact manifold can be approximated by a roughly transitive sequence of graphs. Moreover, there are sequence of roughly transitive graphs with no convergent subsequence, but with a good control on the volume. Here we will sketch the construction of such a sequence.

Recall that compactness is closed under Gromov-Hausdorff limit. Hence a limit X of our sequence X_n, if a limit exists, is necessarily compact. As a result, X has doubling property at any fixed scale: in particular there exists $k \in \mathbb{N}$ such that any ball of radius $\mathrm{diam}(X)/2$ is covered by k balls of radius $\mathrm{diam}(X)/4$. Then for n large enough, balls of radius $\mathrm{diam}(X_n)/2$ are covered by $2k$ balls of radius $\mathrm{diam}(X)/4$.

We start by picking a sequence of Cayley graphs Y_n with no converging subsequence (but without control on the volume). For example, let S be a finite generating subset of $G = \mathrm{SL}(3, \mathbb{Z})$, and let Y_n be the Cayley graph of $G_n = \mathrm{SL}(3, \mathbb{Z}/n\mathbb{Z})$ associated to the (projected) generated set S. The fact that Y_n is an expander violates the previous doubling condition, and hence Y_n does not have any converging subsequence (we leave this easy and standard fact to the reader).

Then convert G_n-equivariantly the Y_n into Riemannian surfaces S_n by replacing edges with empty tubes, and smoothing the joints that correspond to vertices in Y_n. The radius of the tubes in S_n will be some L_n to be determined later, and the length will be $2L_n$. Independently of the choice of L_n, these S_n are uniformly roughly transitive. This follows because the S_n are all rescaled versions of manifolds S'_n that cover the same compact manifold M. For any two points $x, y \in M$, there is a diffeomorphism of M taking x to y, with uniform bounds on the derivative and the derivative of the inverse. This property extends to the covering manifolds S'_n, and thus to S_n.

To obtain the sequence X_n of roughly transitive graphs, replace S_n with a tiling with bounded faces, chosen so that the X_n remain roughly transitive. Choose L_n large enough so that $|X_n| = o(\log(\operatorname{diam}(X_n)) \operatorname{diam}(X_n)^2)$. This is possible, for example, by choosing L_n to grow asymptotically faster than $\frac{1}{D_n^2}(|Y_n| - D_n^2 \log D_n)$, where D_n denotes the diameter of Y_n. Let X'_n denote X_n normalized by $\operatorname{diam}(X_n)$. It remains to show that X'_n has no convergent subsequence. However, the Gromov-Hausdorff distance from X'_n to Y_n is roughly $L_n / \operatorname{diam}(X_n) = 1/\operatorname{diam}(Y_n) \to 0$. Since Y_n has no convergent subsequence, neither does X'_n.

Remark: Note that in the above construction we can bound the volume by a function of the diameter which is as close as we want to quadratic. But this leaves open the problem of finding a sequence of roughly transitive graphs with subquadratic growth which does not admit a converging subsequence. Also we do not know what could be the best δ for which Theorem 2 holds ($\delta = 2$?). Finally finding a *converging* counter-example to Theorem 2 with $\delta < 2$ would be of special interest as the limit would be quite an exotic object: it would be a compact geodesic metric space with Hausdorff dimension in $(1, 2)$, and such that for every pair of points $x, y \in X$, there exists a C-bilipschitz homeomorphism sending x to y, where C only depends on X. We do not know if such object exists.

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